Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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Contents

Definition

Definition

A *matrix of the size* $m \times n$ *over a field* $\mathbb{K}$ (or simply a *matrix*) is a rectangular table of elements $a_{ij} \in \mathbb{K}$ for $i = 1, \ldots, m, j = 1, \ldots, n$:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \ldots & \ldots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

We denote this matrix by $(a_{ij})_{m\times n}$ or simply (a_{ij}). The element a_{ij} is called *ij-entry*. Usually we denote matrices by capital letters $A, B, \ldots$. We say that two matrices A, B are equal and write $A = B$ if they are of the same size and have the same ij -entries. The set of matrices of size $m \times n$ over $\mathbb{K}$ is denoted by $\mathbb{K}^n_m$.

Definition

Definition

Given a matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

the n-tuples $(a_{11}, a_{12}, \dots, a_{1n}) \dots (a_{m1}, a_{m2}, \dots, a_{mn})$ are *rows* and the m-tuples

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix}, \dots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{pmatrix}$$

are *columns*.

Example

The matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ -3 & \frac{1}{2} & 4 \end{pmatrix}$$

is a 2×3 matrix over $\mathbb{R}$.

$$(1, 0, 2), (-3, \frac{1}{2}, 4) \text{ are rows and}$$

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ are columns.}$$

Fact

Let A be an $m \times n$ matrix over a field $\mathbb{K}$. Rows of A are members of the vector space $\mathbb{K}^n$ and columns are members of $\mathbb{K}^m$.

Square matrices

Definition

An $m \times n$ matrix is called *square matrix* if $m = n$. A square matrix is called *symmetric* if $a_{ij} = a_{ji}$ for any $i = 1, \ldots, n$, $j = 1, \ldots, n$.

Example:

$$\begin{pmatrix} 1 & 0 \\ -3 & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

Definition

The n-tuple $(a_{11}, a_{22}, \ldots, a_{nn})$ is called *the main diagonal* of a square matrix $(a_{ij})_{n \times n}$.

Contents

Matrix addition

Let A and B be two matrices of the same size:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ & \dots & \dots & \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

Definition

The *sum of* A and B, denoted by $A + B$, is the matrix

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ & \dots & \dots & \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

Matrix scalar multiplication

Definition

The *product of* a matrix A by a scalar $k \in \mathbb{K}$, denoted by $k \cdot A$, is the matrix

$$k \cdot A = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ & \dots & \dots & \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}.$$

Notation:

$$-A := (-1) \cdot A, \quad A - B := A + (-B).$$

Example

$$\begin{pmatrix} 1 & 0 & 2 \\ -3 & \frac{1}{2} & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 & 4 \\ 2 & \frac{3}{2} & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 6 \\ -1 & 2 & 4 \end{pmatrix}$$

$$3 \cdot \begin{pmatrix} 1 & 0 & 2 \\ -3 & \frac{1}{2} & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 6 \\ -9 & \frac{3}{2} & 12 \end{pmatrix}$$

Warning

Addition of matrices with different sizes is NOT defined.

Properties

Definition

The $m \times n$ matrix with entries all zeros is called *the zero matrix* and is denoted by $0_{m\times n}$ or 0 if the size is clear from the context.

Theorem

Let A, B be $m \times n$ matrices over a field $\mathbb{K}$ and let $k_1, k_2 \in \mathbb{K}$ be two scalars. Then

$$
\begin{aligned}
&A + 0 = 0 + A = A, & k_1(A+B) = k_1 \cdot A + k_1 \cdot B \\
&(A+B)+C = A+(B+C), & (k_1+k_2)A = k_1 \cdot A + k_2 \cdot A \\
&A + B = B + A, & (k_1 \cdot k_2)A = k_1(k_2 \cdot A) \\
&A - A = 0, & 1 \cdot A = A
\end{aligned}
$$

Matrix multiplication

Definition

Let A be a $k \times m$-matrix and B be a $m \times n$-matrix. The product $A \cdot B$ of the matrices A, B is the $k \times n$-matrix $C = (c_{ij})$ whose entries are defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj} \text{ for } i = 1, \ldots, k, \; j = 1, \ldots, n.$$

Example:

$$\begin{array}{cc} & \begin{pmatrix} -3 & 1 & 4 & 2 \\ 2 & \frac{3}{2} & 0 & 3 \\ -3 & 1 & 2 & 1 \end{pmatrix} \\ \begin{pmatrix} -3 & 1 & 4 \\ 2 & 1 & 0 \end{pmatrix} & \color{red}{\begin{pmatrix} -1 & 2\frac{1}{2} & -4 & 1 \\ -4 & 3\frac{1}{2} & 8 & 7 \end{pmatrix}} \end{array}$$

Matrix multiplication

Warning

The product $A \cdot B$ of two matrices A, B is defined ONLY when the number of columns in A is the same as the number of rows in B.

Definition

The square matrix of the size $n \times n$ of the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & \ldots & \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

is called the identity matrix and is denoted by $I_{n \times n}$ or I.

Matrix multiplication: properties

Properties

Let A, A_1 be $k \times m$ matrices, B, B_1 be $m \times n$ matrices and C be an $n \times o$ matrix.

$$\begin{aligned}
(A \cdot B) \cdot C &= A \cdot (B \cdot C), \\
A \cdot I_{m\times m} &= I_{k\times k} \cdot A = A, \\
(A + A_1) \cdot B &= A \cdot B + A_1 \cdot B, \\
A \cdot (B + B_1) &= A \cdot B + A \cdot B_1
\end{aligned}$$

Warning

For $A, B \in \mathbb{K}_n^n$ the multiplication is not commutative in general. I.e. sometimes,

$$A \cdot B \neq B \cdot A.$$

Matrix transposition

Definition

Let $A = (a_{ij})$ be an $m \times n$ matrix. The *transpose* of the matrix A, denoted by A^T is an $n \times m$ matrix defined by

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ & \dots & \dots & \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 0 & 2 \\ -3 & \frac{1}{2} & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{2} \\ 2 & 4 \end{pmatrix}$$

Matrix transposition: properties

Fact

A square matrix A is symmetric iff $A^T = A$.

Properties

Let A, B be matrices of appropriate sizes.

$$(A+B)^T = A^T + B^T,$$
$$(A^T)^T = A,$$
$$(k \cdot A)^T = k \cdot A^T$$
$$(A \cdot B)^T = B^T \cdot A^T.$$

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Definition

Definition

By *elementary row operations* on a matrix A we mean the following

1. **(Row switching)** i-th row and j-th row are interchanged ($R_i \leftrightarrow R_j$),
2. **(Row scaling)** each element in i-th row is multiplied by a *nonzero* scalar $k \in \mathbb{K}$ ($kR_i \to R_i$),
3. **(Row addition)** i-th row is replaced by a sum of i-th row and a multiple of j-th row ($R_i + k \cdot R_j \to R_i$).

Definition

Similarly, we define *elementary column operations*.

Elementary operation: example

$$\begin{pmatrix} -3 & 1 & 4 & 2 \\ 2 & \frac{3}{2} & 0 & 3 \\ -3 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & \frac{3}{2} & 0 & 3 \\ -3 & 1 & 4 & 2 \\ -3 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{3 \cdot R_2}$$

$$\begin{pmatrix} 2 & \frac{3}{2} & 0 & 3 \\ -9 & 3 & 12 & 6 \\ -3 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{2 \cdot R_1 + R_2 \to R_2} \begin{pmatrix} 2 & \frac{3}{2} & 0 & 3 \\ -5 & 6 & 12 & 12 \\ -3 & 1 & 2 & 1 \end{pmatrix}.$$

Properties

Theorem

Each of the elementary row (column) operations is invertible. The inverse is also an elementary row (resp. column) operation.

Proof:

1. The inverse of $R_i \leftrightarrow R_j$ is $R_j \leftrightarrow R_i$,
2. the inverse of $k \cdot R_i \to R_i$ for $k \neq 0$ is $\frac{1}{k} \cdot R_i \to R_i$,
3. the inverse of $R_i + k \cdot R_j \to R_i$ is $R_i - k \cdot R_j \to R_i$.

If we substitute the letter R with C we get the proof for elementary column operations.

Row equivalence

Definition

We say that matrices A and B of the same sizes are *row equivalent* if one can be obtained from the other using elementary row operations.

Example:

$$A = \begin{pmatrix} -3 & 1 & 4 & 2 \\ 2 & \frac{3}{2} & 0 & 3 \\ -3 & 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & \frac{3}{2} & 0 & 3 \\ -5 & 6 & 12 & 12 \\ -3 & 1 & 2 & 1 \end{pmatrix}.$$

Row echelon form

Definition

We say that a matrix A is in *row echelon form* if

1. all zero rows are at the bottom,
2. the first nonzero number in the i-th row is to the right from the first nonzero coefficient in the row above it.

Example:

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 6 & 12 & 12 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row echelon form: properties

Fact

Any matrix A is row equivalent to a matrix in row echelon form.

Example (proof?):

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 4 & 1 & 8 \end{pmatrix} \xrightarrow{R_2 - 4R_1 \to R_3}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Rank of a matrix

Definition

Let A be an $m \times n$ matrix over $\mathbb{K}$. *Row rank* of a matrix A is the dimension of the subspace of $\mathbb{K}^n$ spanned by the rows of A. *Column rank* of the matrix A is the dimension of the subspace of $\mathbb{K}^m$ spanned by the columns of A.

Theorem

For every matrix A the row rank of A is equal to the column rank of A.

Definition

Rank of a matrix A, denoted $r(A)$, is its row rank (or column rank).

Rank: properties and examples

Consider the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$\dim(span(\{(1,2),(2,4)\})) = \dim\{a(1,2)+b(2,4) \mid a,b \in \mathbb{R}\} = \dim\{a(1,2)+2b(1,2) \mid a,b \in \mathbb{R}\} = \dim\{(a+2b)(1,2) \mid a,b \in \mathbb{R}\} = \dim(span(\{(1,2)\})) = 1.$

Hence, $r(A) = 1$.

Theorem

If a matrix A is row equivalent to a matrix B then $r(A) = r(B)$.

Rank and row echelon form

Theorem

Let A be a matrix. The number of non-zero rows in a matrix in row echelon form which is row equivalent to A is equal to rank of A.

Rank and dimension

Theorem

The dimension of a subspace W of a space $\mathbb{K}^n$ spanned by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is equal to the rank of the following matrix:

$$\begin{pmatrix} \mathbf{v}_1 \\ \ldots \\ \mathbf{v}_m \end{pmatrix}$$